

q -Exponentials on quantum spaces

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Abstract. We present explicit formulae for q -exponentials on quantum spaces which could be of particular importance in physics, i.e. the q -deformed Minkowski space and the q -deformed Euclidean space with two, three or four dimensions. Furthermore, these formulae can be viewed as 2-, 3- or 4-dimensional analogues of the well-known q -exponential function.

1 Introduction

In this work we would like to continue our program for developing a non-commutative analysis, which in the following is referred to as q -analysis. One of our motivations for doing this is that a field theory based on q -analysis should be well behaved in the UV-range [1–4]. So far we have concerned ourselves with explicit formulae for star-products [5], representations of partial derivatives [6] as well as q -integrals [7]. All of these mathematical objects have been considered for such non-commutative spaces which could be of particular importance in physics, i.e. the q -deformed Minkowski space and the q -deformed Euclidean space with three or four dimensions. Our goal now is to take a step towards completing this program by providing also explicit formulae for q -exponentials.

Before doing this let us recall some basic aspects of our approach. As already mentioned, q -analysis can be regarded as a non-commutative analysis formulated within the framework of quantum spaces [8–10]. These quantum spaces are defined as comodule algebras of quantum groups and can therefore be interpreted as deformations of ordinary coordinate algebras [11]. For our purposes it is at first sufficient to consider a quantum space as an algebra \mathcal{A}_q of formal power series in the non-commuting coordinates X_1, X_2, \dots, X_n

$$\mathcal{A}_q = \mathbb{C}[[X_1, \dots, X_n]]/\mathcal{I}, \quad (1)$$

where \mathcal{I} denotes the ideal generated by the relations of the non-commuting coordinates. The algebra \mathcal{A}_q satisfies the Poincaré-Birkhoff-Witt property, i.e. the dimension of the subspace of homogeneous polynomials should be the same as for commuting coordinates. This property is the deeper reason why the monomials of normal ordering $X_1 X_2 \dots X_n$ constitute a basis of \mathcal{A}_q . In particular, we can establish a vector space isomorphism between \mathcal{A}_q and the commutative algebra \mathcal{A} generated by ordinary coordinates

x_1, x_2, \dots, x_n :

$$\begin{aligned} \mathcal{W} : \mathcal{A} &\longrightarrow \mathcal{A}_q, \\ \mathcal{W}(x_1^{i_1} \dots x_n^{i_n}) &= X_1^{i_1} \dots X_n^{i_n}. \end{aligned} \quad (2)$$

This vector space isomorphism can be extended to an algebra isomorphism introducing a non-commutative product in \mathcal{A} , the so-called \star -product [12, 13]. This product is defined by the relation

$$\mathcal{W}(f \star g) = \mathcal{W}(f) \cdot \mathcal{W}(g), \quad (3)$$

where f and g are formal power series in \mathcal{A} . Additionally, for each quantum space exists a symmetry algebra [14, 15] and a covariant differential calculus [16] which can provide an action upon the quantum spaces under consideration. By means of the relation

$$\mathcal{W}(h \triangleright f) := h \triangleright \mathcal{W}(f), \quad h \in \mathcal{H}, f \in \mathcal{A}, \quad (4)$$

we are also able to introduce an action upon the corresponding commutative algebra.

To gain further insight it is also useful to consider quantum spaces from a point of view provided by category theory. A category for our purposes is just a collection of objects X, Y, Z, \dots and a set $\text{Mor}(X, Y)$ of morphisms between two objects X, Y such that a composition of morphisms is defined which has similar properties to the composition of maps. In particular, we are interested in tensor categories. These are categories that have a product, denoted \otimes and called the tensor product, which admits several “natural” properties such as associativity and existence of a unit object. For a more formal treatment we refer the interested reader to the presentations in [17–19].

Essential for us is the fact that the representations (quantum spaces) of a given quasitriangular Hopf algebra (quantum algebra) are the objects of a tensor category, if the action of the Hopf algebra on the tensor product of

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two quantum spaces is defined by¹

$$h \triangleright (v \otimes w) = (h_{(1)} \triangleright v) \otimes (h_{(2)} \triangleright w). \tag{5}$$

In this category exist a number of morphisms of particular importance. First of all, for any pair of objects X, Y there is an isomorphism $\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ such that $(g \otimes f) \circ \Psi_{X,Y} = \Psi_{X',Y'} \circ (f \otimes g)$ for arbitrary morphisms $f \in \text{Mor}(X, X')$ and $g \in \text{Mor}(Y, Y')$ and the hexagon axiom holds. This hexagon axiom is the validity of the two conditions

$$\Psi_{X,Y} \circ \Psi_{Y,Z} = \Psi_{X \otimes Y, Z}, \quad \Psi_{X,Z} \circ \Psi_{X,Y} = \Psi_{X, Y \otimes Z}. \tag{6}$$

A tensor category having the above property is called a braided tensor category. Furthermore, for any algebra B in this category there are morphisms $\Delta : B \rightarrow B \otimes B$, $S : B \rightarrow B$ and $\varepsilon : B \rightarrow \mathbb{C}$ forming a braided Hopf algebra, i.e. Δ, S and ε obey the usual axioms of a Hopf algebra, but now as morphisms in the braided category. These considerations show that under suitable assumptions our quantum spaces can be viewed as braided Hopf algebras. For a deeper understanding of these ideas we also refer the reader to the excellent presentations in [20, 21].

Now let us make contact with another very important ingredient of our braided tensor category. For this purpose we suppose that our category is equipped with dual objects B^* for each algebra B in the category. This means that we have a dual pairing

$$\langle , \rangle : B \otimes B^* \rightarrow K \quad \text{with} \quad \langle e_a, f^b \rangle = \delta_a^b, \tag{7}$$

where $\{e_a\}$ is a basis in B and $\{f_a\}$ a dual basis in B^* . This allows us to introduce an exponential which from an abstract point of view is nothing other than an object whose dualization is the evaluation map [9]. Thus, the exponential is given by the map²

$$\exp : K \rightarrow B^* \otimes B \quad \text{with} \quad \exp = \sum_a f^a \otimes e_a \tag{8}$$

and satisfies the following relations

$$\begin{aligned} (\Delta \otimes \text{id}) \exp &= \sum_{j,k} f^j \otimes f^k \otimes e_k e_j = \text{exp}_{23} \text{exp}_{13}, \\ (\text{id} \otimes \Delta) \exp &= \sum_{j,k} f^j f^k \otimes e_k \otimes e_j = \text{exp}_{13} \text{exp}_{12}. \end{aligned} \tag{9}$$

To make this concrete, we recall that it was shown in [22] that there is such a duality pairing of quantum space coordinates and the corresponding partial derivatives. Explicitly, we have

$$\begin{aligned} \langle , \rangle : \mathcal{M}_\partial \otimes \mathcal{M}_x &\rightarrow K \\ \text{with} \quad \langle f(\partial_i), g(x^j) \rangle &\equiv \varepsilon(f(\partial_i) \triangleright g(x^j)). \end{aligned} \tag{10}$$

¹ We write the coproduct in the so-called Sweedler notation, i.e. $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

² In the quantum group case such an object is often referred to as the canonical element.

As it is our aim to derive explicit formulae for q -exponentials of q -deformed Minkowski space and q -deformed Euclidean space with up to four dimensions, our task is therefore to determine a basis of the coordinate algebra \mathcal{M}_x being dual to a given one of the derivative algebra \mathcal{M}_∂ . Inserting the elements of these two bases into formula (8) will then provide us with explicit expressions for the exponentials. It should be stressed that the existence of the algebra isomorphism \mathcal{W} defined in (2) enables us to carry out all the necessary calculations in the corresponding commutative algebras. In doing so we are led to 2-, 3- and 4-dimensional analogues of the well-known q -exponential function [23, 24].

As it was shown in [6], the partial derivatives can act on the algebra of quantum space coordinates in four different ways, i.e. by right actions, left actions and their conjugated counterparts. For this reason there are four possibilities for defining a pairing between coordinates and derivatives. More concretely, we can distinguish the pairings

$$\begin{aligned} \langle f(\partial), g(x) \rangle_{L, \hat{R}} &\equiv \varepsilon(f(\partial) \triangleright g(x)), \\ \langle f(\hat{\partial}), g(x) \rangle_{\hat{L}, R} &\equiv \varepsilon(f(\hat{\partial}) \bar{\triangleright} g(x)), \\ \langle f(x), g(\partial) \rangle_{L, \hat{R}} &\equiv \varepsilon(f(x) \bar{\triangleleft} g(\partial)), \\ \langle f(x), g(\hat{\partial}) \rangle_{\hat{L}, R} &\equiv \varepsilon(f(x) \triangleleft g(\hat{\partial})), \end{aligned} \tag{11}$$

where $\hat{\partial}^A$ differs from ∂^A by a normalization factor, only [6, 25, 26]. Clearly, each of these pairings will lead to its own exponential. It should also be clear from the above considerations that the different exponentials can be linked via the same crossing symmetries which have already helped us in [6] to transform the underlying representations of partial derivatives into each other.

There are two properties of the considered exponentials worth recording here. First of all, the exponentials are normalized in such a way that

$$\begin{aligned} (\varepsilon \otimes \text{id}) \exp(x \mid \partial) &= 1, \\ (\text{id} \otimes \varepsilon) \exp(x \mid \partial) &= 1. \end{aligned} \tag{12}$$

These equalities result from

$$\begin{aligned} (\text{id} \otimes \varepsilon) \exp &= \sum_a f^a \otimes \varepsilon(e_a) \\ &= \sum_a f^a \otimes \langle e_a, 1 \rangle \\ &= 1 \otimes 1 = 1, \end{aligned} \tag{13}$$

$$\begin{aligned} (\varepsilon \otimes \text{id}) \exp &= \sum_a \varepsilon(f^a) \otimes e_a \\ &= \sum_a \langle 1, f^a \rangle \otimes e_a \\ &= 1 \otimes 1 = 1. \end{aligned} \tag{14}$$

Second the exponentials obey the identities [21]

$$\begin{aligned} \partial^i \triangleright \exp(x_{\hat{R}} \mid \partial_L) &= \exp(x_{\hat{R}} \mid \partial_L) \star \partial^i, \\ \hat{\partial}^i \bar{\triangleright} \exp(x_R \mid \hat{\partial}_{\hat{L}}) &= \exp(x_R \mid \hat{\partial}_{\hat{L}}) \star \hat{\partial}^i, \end{aligned}$$

$$\begin{aligned} \partial^i \star \exp(\partial_{\hat{R}} | x_L) &= \exp(\partial_{\hat{R}} | x_L) \bar{\triangleright} \partial^i, \\ \hat{\partial}^i \star \exp(\hat{\partial}_R | x_{\hat{L}}) &= \exp(\hat{\partial}_R | x_{\hat{L}}) \triangleleft \hat{\partial}^i \end{aligned} \quad (15)$$

which tell us that our exponentials can be regarded as q -analogues of classical plane-waves. For a proof of these formulae one has to realize that an algebra acts on its dual via

$$e_a \triangleright f^b = \langle e_a, f_{(1)}^b \rangle f_{(2)}^b. \quad (16)$$

With this relation at hand one can proceed in the following fashion:

$$\begin{aligned} (e_b \otimes 1) \triangleright \exp &= \sum_a e_b \triangleright f^a \otimes e_a \\ &= \sum_a \langle e_b, f_{(1)}^a \rangle f_{(2)}^a \otimes e_a \\ &= \sum_{a,c} \langle e_b, f^a \rangle f^c \otimes e_c e_a \\ &= \sum_c f^c \otimes e_c e_b \\ &= \exp \cdot (1 \otimes e_b). \end{aligned} \quad (17)$$

Notice that the third equality uses the first relation in (9). Through a slight modification to these arguments, one can prove the corresponding identities concerning right actions.

2 2-dimensional q -deformed Euclidean space

The q -deformed Euclidean space with two dimensions is generated by coordinates X^i , $i = 1, 2$, subject to the relation

$$X^1 X^2 = q X^2 X^1. \quad (18)$$

As it is well-known, there are two covariant differential calculi on this quantum space given by [16]

$$\begin{aligned} \partial^i X^j &= \varepsilon^{ij} + q^2 (\hat{R}^{-1})_{kl}^{ij} X^k \partial^l, \\ \hat{\partial}^i X^j &= \varepsilon^{ij} + q^{-2} (\hat{R})_{kl}^{ij} X^k \partial^l, \quad i, j = 1, 2, \end{aligned} \quad (19)$$

where \hat{R} and ε^{ij} denote respectively the R -matrix of the quantum algebra $U_q(su_2)$ and the corresponding quantum metric³. Written out, we get

$$\partial_X^1 X^1 = q X^1 \partial_X^1, \quad (20)$$

$$\partial_X^1 X^2 = -q^{-\frac{1}{2}} + q^2 X^2 \partial_X^1,$$

$$\partial_X^2 X^1 = q^{\frac{1}{2}} + q^2 X^1 \partial_X^2 - q^2 \lambda X^2 \partial_X^1, \quad (21)$$

$$\partial_X^2 X^2 = q X^2 \partial_X^2$$

and

$$\hat{\partial}_X^1 X^1 = q^{-1} X^1 \hat{\partial}_X^1,$$

³ Our notation, conventions and definitions are listed in the appendix.

$$\hat{\partial}_X^1 X^2 = q^{-\frac{1}{2}} + q^{-2} X^2 \hat{\partial}_X^1 + q^{-2} \lambda X^1 \hat{\partial}_X^2, \quad (22)$$

$$\hat{\partial}_X^2 X^1 = -q^{\frac{1}{2}} + q^{-2} X^1 \hat{\partial}_X^2,$$

$$\hat{\partial}_X^2 X^2 = q^{-1} X^2 \hat{\partial}_X^2. \quad (23)$$

with $\lambda = q - q^{-1}$. From these commutation relations one can calculate the action of the partial derivatives on monomials of a given normal ordering. In this way we have derived the expressions

$$\begin{aligned} \partial^1 \triangleright (X^2)^{m_2} (X^1)^{m_1} &= -q^{-\frac{1}{2}} [[m_2]]_{q^2} (X^2)^{m_2-1} (X^1)^{m_1}, \\ \partial^2 \triangleright (X^2)^{m_2} (X^1)^{m_1} &= q^{\frac{1}{2}+m_2} [[m_1]]_{q^2} (X^2)^{m_2} (X^1)^{m_1-1} \end{aligned} \quad (24)$$

and

$$\begin{aligned} \hat{\partial}^1 \bar{\triangleright} (X^1)^{m_1} (X^2)^{m_2} &= q^{-\frac{1}{2}-m_1} [[m_2]]_{q^{-2}} (X^1)^{m_1} (X^2)^{m_2-1}, \\ \hat{\partial}^2 \bar{\triangleright} (X^1)^{m_1} (X^2)^{m_2} &= -q^{\frac{1}{2}} [[m_1]]_{q^{-2}} (X^1)^{m_1-1} (X^2)^{m_2}, \end{aligned} \quad (25)$$

where the antisymmetric q -number is defined by [27]

$$[[c]]_{q^a} \equiv \frac{1 - q^{ac}}{1 - q^a}, \quad a, c \in \mathbb{C}. \quad (26)$$

Iteration of (24) and (25) then leads to the formulae for the dual pairing⁴

$$\begin{aligned} &\left\langle (\partial^2)^{n_2} (\partial^1)^{n_1}, (X^2)^{m_2} (X^1)^{m_1} \right\rangle_{L, \hat{R}} \\ &= \varepsilon \left((\partial^2)^{n_2} (\partial^1)^{n_1} \triangleright (X^2)^{m_2} (X^1)^{m_1} \right) \\ &= \delta_{m_1, n_2} \delta_{m_2, n_1} \left(-q^{-\frac{1}{2}} \right)^{n_1} q^{\frac{1}{2}n_2} [[n_1]]_{q^2}! [[n_2]]_{q^2}!, \end{aligned} \quad (27)$$

$$\begin{aligned} &\left\langle (\hat{\partial}^1)^{n_1} (\hat{\partial}^2)^{n_2}, (X^1)^{m_1} (X^2)^{m_2} \right\rangle_{\hat{L}, R} \\ &= \varepsilon \left((\hat{\partial}^1)^{n_1} (\hat{\partial}^2)^{n_2} \bar{\triangleright} (X^1)^{m_1} (X^2)^{m_2} \right) \\ &= \delta_{m_1, n_2} \delta_{m_2, n_1} \left(-q^{\frac{1}{2}} \right)^{n_2} \left(q^{-\frac{1}{2}} \right)^{n_1} [[n_1]]_{q^{-2}}! [[n_2]]_{q^{-2}}!, \end{aligned} \quad (28)$$

where the q -factorials are given by

$$\begin{aligned} [[m]]_{q^a}! &\equiv [[1]]_{q^a} [[2]]_{q^a} \cdots [[m]]_{q^a}, \\ [[0]]_{q^a}! &\equiv 1. \end{aligned} \quad (29)$$

The above results can be simplified further by introducing partial derivatives with lower indices defined by

$$\partial_i = \varepsilon_{ij} \partial^j, \quad \hat{\partial}_i = \varepsilon_{ij} \hat{\partial}^j. \quad (30)$$

Thus, we end up with

⁴ Notice that $\varepsilon(f(\underline{x})) = f(\underline{x} = 0)$.

$$\begin{aligned} & \left\langle (\partial_1)^{n_1} (\partial_2)^{n_2}, (X^2)^{m_2} (X^1)^{m_1} \right\rangle_{L, \hat{R}} \\ &= \delta_{m_1, n_1} \delta_{m_2, n_2} [[n_1]]_{q^2}! [[n_2]]_{q^2}!, \end{aligned} \tag{31}$$

$$\begin{aligned} & \left\langle (\hat{\partial}_2)^{n_2} (\hat{\partial}_1)^{n_1}, (X^1)^{m_1} (X^2)^{m_2} \right\rangle_{\hat{L}, R} \\ &= \delta_{m_1, n_1} \delta_{m_2, n_2} [[n_1]]_{q^{-2}}! [[n_2]]_{q^{-2}}!. \end{aligned} \tag{32}$$

These expressions enable us to identify the two sets of basis elements being dual to each other. With this knowledge we are now in a position to apply formula (6) giving us

$$\widetilde{\text{exp}}(x_{\hat{R}} | \partial_L) = \sum_{n_1, n_2=0}^{\infty} \frac{(X^2)^{n_2} (X^1)^{n_1} \otimes (\partial_1)^{n_1} (\partial_2)^{n_2}}{[[n_1]]_{q^2}! [[n_2]]_{q^2}!}, \tag{33}$$

$$\text{exp}(x_R | \hat{\partial}_{\hat{L}}) = \sum_{n_1, n_2=0}^{\infty} \frac{(X^1)^{n_1} (X^2)^{n_2} \otimes (\hat{\partial}_2)^{n_2} (\hat{\partial}_1)^{n_1}}{[[n_1]]_{q^{-2}}! [[n_2]]_{q^{-2}}!}, \tag{34}$$

where the tilde in the first formula shall remind us of the fact that this exponential compared to the second one refers to a different choice for the normal ordering of the coordinates and derivatives.

The remainder of this section is devoted to the calculation of exponentials corresponding to right representations of partial derivatives. For this purpose it is useful to realize that left representations are transformed to right ones by conjugation, i.e.

$$\begin{aligned} \overline{\partial \triangleright f} &= \bar{f} \bar{\triangleleft} \bar{\partial}, \\ \overline{\partial \triangleright f} &= \bar{f} \bar{\triangleleft} \bar{\partial}. \end{aligned} \tag{35}$$

In view of this relationship and the conjugation properties⁵

$$\overline{h^i} = \varepsilon_{ij} h^j, \tag{36}$$

where h^i stands for X^i or ∂^i , the right actions of partial derivatives on normally ordered monomials take on the form

$$\begin{aligned} & (X^2)^{m_2} (X^1)^{m_1} \bar{\triangleleft} \partial^1 \\ &= -q^{\frac{1}{2} + m_1} [[m_2]]_{q^2} (X^2)^{m_2 - 1} (X^1)^{m_1}, \\ & (X^2)^{m_2} (X^1)^{m_1} \bar{\triangleleft} \partial^2 \\ &= q^{-\frac{1}{2}} [[m_1]]_{q^2} (X^2)^{m_2} (X^1)^{m_1 - 1} \end{aligned} \tag{37}$$

and

$$\begin{aligned} & (X^1)^{m_1} (X^2)^{m_2} \triangleleft \hat{\partial}^1 \\ &= q^{\frac{1}{2}} [[m_2]]_{q^{-2}} (X^1)^{m_1} (X^2)^{m_2 - 1}, \\ & (X^1)^{m_1} (X^2)^{m_2} \triangleleft \hat{\partial}^2 \\ &= -q^{-\frac{1}{2} - m_2} [[m_1]]_{q^{-2}} (X^1)^{m_1 - 1} (X^2)^{m_2}. \end{aligned} \tag{38}$$

With the very same reasonings already applied to left representations we can show that

⁵ We are here following the approach of [28] which implies that partial derivatives and coordinates obey the same conjugation properties.

$$\begin{aligned} & \left\langle (X_1)^{m_1} (X_2)^{m_2}, (\partial^2)^{n_2} (\partial^1)^{n_1} \right\rangle_{L, \hat{R}} \\ &= \varepsilon \left((X_1)^{m_1} (X_2)^{m_2} \bar{\triangleleft} (\partial^2)^{n_2} (\partial^1)^{n_1} \right) \\ &= \delta_{m_1, n_1} \delta_{m_2, n_2} [[n_1]]_{q^2}! [[n_2]]_{q^2}!, \end{aligned} \tag{39}$$

$$\begin{aligned} & \left\langle (X_2)^{m_2} (X_1)^{m_1}, (\hat{\partial}^1)^{n_1} (\hat{\partial}^2)^{n_2} \right\rangle_{\hat{L}, R} \\ &= \varepsilon \left((X_2)^{m_2} (X_1)^{m_1} \triangleleft (\hat{\partial}^1)^{n_1} (\hat{\partial}^2)^{n_2} \right) \\ &= \delta_{m_1, n_1} \delta_{m_2, n_2} [[n_1]]_{q^{-2}}! [[n_2]]_{q^{-2}}!, \end{aligned} \tag{40}$$

which, in turn, leads to

$$\widetilde{\text{exp}}(\partial_{\hat{R}} | x_L) = \sum_{n_1, n_2=0}^{\infty} \frac{(\partial^2)^{n_2} (\partial^1)^{n_1} \otimes (X_1)^{n_1} (X_2)^{n_2}}{[[n_1]]_{q^2}! [[n_2]]_{q^2}!}, \tag{41}$$

$$\text{exp}(\hat{\partial}_R | x_{\hat{L}}) = \sum_{n_1, n_2=0}^{\infty} \frac{(\hat{\partial}^1)^{n_1} (\hat{\partial}^2)^{n_2} \otimes (X_2)^{n_2} (X_1)^{n_1}}{[[n_1]]_{q^{-2}}! [[n_2]]_{q^{-2}}!}, \tag{42}$$

where we have introduced coordinates with lower indices by setting

$$X_i = \varepsilon_{ij} X^j. \tag{43}$$

It is now obvious, from what we have done so far, that the different exponentials can be transformed into each other by applying some simple rules. First of all, we can verify the existence of a correspondence given by

$$\begin{aligned} \widetilde{\text{exp}}(x_{\hat{R}} | \partial_L) &\stackrel{i \rightarrow i'}{q \leftarrow 1/q} \text{exp}(x_R | \hat{\partial}_{\hat{L}}), \\ \widetilde{\text{exp}}(\partial_{\hat{R}} | x_L) &\stackrel{i \rightarrow i'}{q \leftarrow 1/q} \text{exp}(\hat{\partial}_R | x_{\hat{L}}), \end{aligned} \tag{44}$$

where the symbol $\stackrel{i \rightarrow i'}{q \leftarrow 1/q}$ indicates a transition via one of the following two substitutions:

$$q \leftrightarrow q^{-1}, \quad \partial_i \leftrightarrow \hat{\partial}_{i'}, \quad X^i \leftrightarrow X^{i'}, \tag{45}$$

$$q \leftrightarrow q^{-1}, \quad \partial^i \leftrightarrow \hat{\partial}^{i'}, \quad X_i \leftrightarrow X_{i'}, \tag{46}$$

with $i' = 3 - i$. Likewise, one can read off the transformation rules

$$\begin{aligned} \widetilde{\text{exp}}(x_{\hat{R}} | \partial_L) &\stackrel{i \leftrightarrow i'}{\leftarrow} \widetilde{\text{exp}}(\partial_{\hat{R}} | x_L), \\ \text{exp}(x_R | \hat{\partial}_{\hat{L}}) &\stackrel{i \leftrightarrow i'}{\leftarrow} \text{exp}(\hat{\partial}_R | x_{\hat{L}}), \end{aligned} \tag{47}$$

where $\stackrel{i \leftrightarrow i'}{\leftarrow}$ now denotes that one can make a transition between the two expressions by applying one of the following two substitutions:

$$X^i \leftrightarrow \partial^i, \quad \partial_i \leftrightarrow X_i, \tag{48}$$

$$X^i \leftrightarrow \hat{\partial}^i, \quad \hat{\partial}_i \leftrightarrow X_i. \tag{49}$$

3 3-dimensional q -deformed Euclidean space

All considerations of the previous section carry over to the q -deformed Euclidean space with three dimensions⁶.

⁶ For a definition of 3-dimensional q -deformed Euclidean space see Appendix A.

Thus, we limit ourselves to stating the results. As in the 2-dimensional case, there are two different covariant differential calculi which are completely described by the commutation relations

$$\begin{aligned} \partial^A X^B &= g^{AB} + (\hat{R}^{-1})_{CD}^{AB} X^C \partial^D, \\ \hat{\partial}^A X^B &= g^{AB} + (\hat{R})_{CD}^{AB} X^C \hat{\partial}^D, \quad A, B \in \{3, +, -\}, \end{aligned} \quad (50)$$

where \hat{R} denotes the R -matrix of the quantum group $SO_q(3)$ and g^{AB} the corresponding quantum metric. In what follows, we restrict attention to the first relation in (50) from which we have derived in [6] the following expressions:

$$\begin{aligned} \partial^- \triangleright (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-} \\ = -q^{-1} [[m_+]_{q^4} (X^+)^{m_+-1} (X^3)^{m_3} (X^-)^{m_-}, \end{aligned} \quad (51)$$

$$\begin{aligned} \partial^3 \triangleright (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-} \\ = q^{2m_+} [[m_3]_{q^2} (X^+)^{m_+-1} (X^3)^{m_3-1} (X^-)^{m_-}, \end{aligned} \quad (52)$$

$$\begin{aligned} \partial^+ \triangleright (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-} \\ = -q^{2m_3+1} [[m_-]_{q^4} (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_--1} \\ - q\lambda [[m_3]_{q^2} [[m_3 - 1]_{q^2} (X^+)^{m_++1} (X^3)^{m_3-2} (X^-)^{m_-}. \end{aligned} \quad (53)$$

Using these formulae we obtain after some tedious steps

$$\begin{aligned} \langle (\partial^+)^{n_+} (\partial^3)^{n_3} (\partial^-)^{n_-}, (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-} \rangle_{L, \hat{R}} \\ = \varepsilon ((\partial^+)^{n_+} (\partial^3)^{n_3} (\partial^-)^{n_-} \triangleright (X^+)^{m_+} (X^3)^{m_3} (X^-)^{m_-}) \\ = \delta_{m_+, n_-} \delta_{m_3, n_3} \delta_{m_-, n_+} \\ \times (-q)^{n_+ - n_-} [[m_+]_{q^4}! [[m_3]_{q^2}! [[m_-]_{q^4}!. \end{aligned} \quad (54)$$

Now, we are again in a position to read off the two sets of basis elements being dual to each other. Finally, this enables us along with

$$\partial_A = g_{AB} \partial^B \quad (55)$$

to write down the exponential as

$$\begin{aligned} \exp(x_{\hat{R}} | \partial_L) \\ = \sum_{\underline{n}=0}^{\infty} \frac{(X^+)^{n_+} (X^3)^{n_3} (X^-)^{n_-} \otimes (\partial_-)^{n_-} (\partial_3)^{n_3} (\partial_+)^{n_+}}{[[n_+]_{q^4}! [[n_3]_{q^2}! [[n_-]_{q^4}!}. \end{aligned} \quad (56)$$

Repeating the identical steps as before, we can also compute explicit formulae for the other types of q -exponentials. These calculations show us the existence of a correspondence given by

$$\begin{aligned} \exp(x_{\hat{R}} | \partial_L) \stackrel{\pm \rightarrow \mp}{q \leftarrow \frac{1}{q}} \widetilde{\exp}(x_R | \hat{\partial}_{\hat{L}}), \\ \exp(\partial_{\hat{R}} | x_L) \stackrel{\pm \rightarrow \mp}{q \leftarrow \frac{1}{q}} \widetilde{\exp}(\hat{\partial}_R | x_{\hat{L}}), \end{aligned} \quad (57)$$

where the symbol $\stackrel{\pm \rightarrow \mp}{q \leftarrow \frac{1}{q}}$ indicates a transition via one of the following two substitutions:

$$q \leftrightarrow q^{-1}, \quad \partial_{\pm} \leftrightarrow \hat{\partial}_{\mp}, \quad \partial_3 \leftrightarrow \hat{\partial}_3, \quad X^{\pm} \leftrightarrow X^{\mp}, \quad (58)$$

$$q \leftrightarrow q^{-1}, \quad \partial^{\pm} \leftrightarrow \hat{\partial}^{\mp}, \quad \partial^3 \leftrightarrow \hat{\partial}^3, \quad X_{\pm} \leftrightarrow X_{\mp}. \quad (59)$$

Additionally, one can verify the transformation rules

$$\begin{aligned} \exp(x_{\hat{R}} | \partial_L) \stackrel{\pm \leftrightarrow \mp}{\leftarrow} \exp(\partial_{\hat{R}} | x_L), \\ \widetilde{\exp}(x_R | \hat{\partial}_{\hat{L}}) \stackrel{\pm \leftrightarrow \mp}{\leftarrow} \widetilde{\exp}(\hat{\partial}_R | x_{\hat{L}}), \end{aligned} \quad (60)$$

where the symbol $\stackrel{\pm \leftrightarrow \mp}{\leftarrow}$ denotes that one can make a transition between the two expressions by applying one of the following two substitutions:

$$X^A \leftrightarrow \partial^A, \quad \partial_A \leftrightarrow X_A, \quad (61)$$

$$X^A \leftrightarrow \hat{\partial}^A, \quad \hat{\partial}_A \leftrightarrow X_A. \quad (62)$$

4 4-dimensional q -deformed Euclidean space

The 4-dimensional Euclidean space [29] (for its definition see Appendix A) can be treated along the same line of arguments as the 2- and 3-dimensional one. Again we begin by considering the commutation relations between partial derivatives and coordinates, which for the two covariant differential calculi read

$$\begin{aligned} \partial^i X^j &= g^{ij} + q(\hat{R}^{-1})_{kl}^{ij} X^k \partial^l, \\ \hat{\partial}^i X^j &= g^{ij} + q^{-1}(\hat{R})_{kl}^{ij} X^k \hat{\partial}^l, \quad i, j = 1, \dots, 4, \end{aligned} \quad (63)$$

with \hat{R} and g^{ij} being the R -matrix of the quantum group $SO_q(4)$ and the corresponding metric, respectively. From the second relation in (63) we have found in [6] the following formulae for the action of partial derivatives on normally ordered monomials:

$$\begin{aligned} \hat{\partial}^1 \bar{\triangleright} (X^{1, \dots, 4})^{\underline{m}} \\ = q^{-1-m_2-m_3} [[m_4]_{q^{-2}} (X^{1, \dots, 4})^{\underline{m}+(0,0,0,-1)} \\ + q^{-1}\lambda [[m_2]_{q^{-2}} [[m_3]_{q^{-2}} (X^{1, \dots, 4})^{\underline{m}+(1,-1,-1,0)}, \\ \hat{\partial}^2 \bar{\triangleright} (X^{1, \dots, 4})^{\underline{m}} = q^{-m_1} [[m_3]_{q^{-2}} (X^{1, \dots, 4})^{\underline{m}+(0,0,-1,0)}, \\ \hat{\partial}^3 \bar{\triangleright} (X^{1, \dots, 4})^{\underline{m}} = q^{-m_1} [[m_2]_{q^{-2}} (X^{1, \dots, 4})^{\underline{m}+(0,-1,0,0)}, \\ \hat{\partial}^4 \bar{\triangleright} (X^{1, \dots, 4})^{\underline{m}} = q[[m_1]_{q^{-2}} (X^{1, \dots, 4})^{\underline{m}+(-1,0,0,0)}, \end{aligned} \quad (64)$$

where, for compactness, we have introduced a new notation:

$$\begin{aligned} (X^{1, \dots, 4})^{\underline{m}} &= (X^1)^{m_1} (X^2)^{m_2} (X^3)^{m_3} (X^4)^{m_4}, \\ (\hat{\partial}_{4, \dots, 1})^{\underline{n}} &= (\hat{\partial}_4)^{n_4} (\hat{\partial}_3)^{n_3} (\hat{\partial}_2)^{n_2} (\hat{\partial}_1)^{n_1}. \end{aligned} \quad (65)$$

Switching to partial derivatives with lower indices,

$$\hat{\partial}_i = g_{ij} \hat{\partial}^j, \quad (66)$$

these expressions imply for the dual pairing the identity

$$\begin{aligned} \langle (\hat{\partial}_{4, \dots, 1})^{\underline{n}}, (X^{1, \dots, 4})^{\underline{m}} \rangle_{\hat{L}, R} \\ = \delta_{m_1, n_1} \delta_{m_2, n_2} \delta_{m_3, n_3} \delta_{m_4, n_4} \\ \cdot [[m_1]_{q^{-2}}! [[m_2]_{q^{-2}}! [[m_3]_{q^{-2}}! [[m_4]_{q^{-2}}!, \end{aligned} \quad (67)$$

which, with the same reasonings as in the previous sections, leads to

$$\begin{aligned} & \exp(x_R | \hat{\partial}_{\hat{L}}) \\ &= \sum_{n=0}^{\infty} \frac{(X^{1,\dots,4})^n \otimes (\hat{\partial}_{4,\dots,1})^n}{[[n_1]]_{q^{-2}}! [[n_2]]_{q^{-2}}! [[n_3]]_{q^{-2}}! [[n_4]]_{q^{-2}}!}. \end{aligned} \quad (68)$$

In complete analogy to the 2- and 3-dimensional case there is again a correspondence between the different types of q -exponentials. First of all, we have

$$\begin{aligned} & \exp(x_R | \hat{\partial}_{\hat{L}}) \xrightarrow{q \leftrightarrow q^{-1}, i \leftrightarrow i'} \widetilde{\exp}(x_{\hat{R}} | \partial_L), \\ & \widetilde{\exp}(\partial_{\hat{R}} | x_L) \xrightarrow{q \leftrightarrow q^{-1}, i \leftrightarrow i'} \exp(\hat{\partial}_R | x_{\hat{L}}), \end{aligned} \quad (69)$$

which concretely means that the expressions on the right- and left-hand side can be transformed into each other by one of the following two substitutions:

$$q \leftrightarrow q^{-1}, \quad \partial_i \leftrightarrow \hat{\partial}_{i'}, \quad X^i \leftrightarrow X^{i'}, \quad (70)$$

$$q \leftrightarrow q^{-1}, \quad \hat{\partial}^i \leftrightarrow \partial^{i'}, \quad X_i \leftrightarrow X_{i'}, \quad (71)$$

where $i = 1, \dots, 4$, and $i' = 5 - i$. In complete analogy to the 2- and 3-dimensional case we can also find the transformations

$$\begin{aligned} & \widetilde{\exp}(x_{\hat{R}} | \partial_L) \xleftrightarrow{i \leftrightarrow i'} \widetilde{\exp}(\partial_{\hat{R}} | x_L), \\ & \exp(x_R | \hat{\partial}_{\hat{L}}) \xleftrightarrow{i \leftrightarrow i'} \exp(\hat{\partial}_R | x_{\hat{L}}) \end{aligned} \quad (72)$$

symbolizing a transition via one of the following two substitutions:

$$X^i \leftrightarrow \partial^i, \quad \partial_i \leftrightarrow X_i, \quad (73)$$

$$X^i \leftrightarrow \hat{\partial}^i, \quad \hat{\partial}_i \leftrightarrow X_i. \quad (74)$$

5 q -deformed Minkowski space

From a physical point of view the most important case we want to discuss in this article is q -deformed Minkowski space [30–32]⁷. There are again two covariant differential calculi given by

$$\begin{aligned} \partial^\mu X^\nu &= g^{\mu\nu} + q^{-2} (\hat{R}_{II}^{-1})_{\rho\sigma}^{\mu\nu} X^\rho \partial^\sigma, \\ \hat{\partial}^\mu X^\nu &= g^{\mu\nu} + q^2 (\hat{R}_{II})_{\rho\sigma}^{\mu\nu} X^\rho \hat{\partial}^\sigma, \\ & \mu, \nu \in \{\pm, 0, 3\}, \end{aligned} \quad (75)$$

where \hat{R}_{II} stands for one of the two R -matrices of the q -deformed Lorentz algebra [35] and $g^{\mu\nu}$ for the corresponding quantum metric. From the above relations we have calculated in [6] representations for partial derivatives. However, the complexity of these representations makes it rather difficult to deduce for the dual pairing

⁷ For its definition see Appendix A. Other versions of q -deformed Minkowski space are given in [33,34].

a closed expression from which we could read off the two sets of basis elements. Thus, we cannot directly apply the procedure of the last two sections for determining a basis being dual to a given one of normally ordered monomials. For this reason we would like to present a different method for calculating q -exponentials.

To begin, our first job is now to seek a useful ansatz describing the q -exponentials. Since our exponentials are required to be bosonic [36] they have to satisfy the properties

$$\begin{aligned} \Lambda \triangleright (\exp(x_R | \hat{\partial}_{\hat{L}})) &= (\Lambda \otimes \Lambda) \triangleright \exp(x_R | \hat{\partial}_{\hat{L}}) \\ &= \varepsilon(\Lambda) \exp(x_R | \hat{\partial}_{\hat{L}}) = \exp(x_R | \hat{\partial}_{\hat{L}}), \end{aligned} \quad (76)$$

$$\begin{aligned} \tau^3 \triangleright (\exp(x_R | \hat{\partial}_{\hat{L}})) &= (\tau^3 \otimes \tau^3) \triangleright \exp(x_R | \hat{\partial}_{\hat{L}}) \\ &= \varepsilon(\tau^3) \exp(x_R | \hat{\partial}_{\hat{L}}) = \exp(x_R | \hat{\partial}_{\hat{L}}), \end{aligned} \quad (77)$$

with τ^3 being a grouplike generator of the q -Lorentz algebra and Λ denoting the associated scaling operator [37]. Recalling that

$$\begin{aligned} \Lambda \triangleright X^\mu &= q^{-2} X^\mu, \\ \Lambda \triangleright \hat{\partial}^\mu &= q^2 \hat{\partial}^\mu, \quad \mu \in \{+, 3, 0, -\}, \end{aligned} \quad (78)$$

and

$$\begin{aligned} \tau^3 \triangleright X^\pm &= q^{\mp 4} X^\pm, \quad \tau^3 \triangleright X^0 = X^0, \quad \tau^3 \triangleright X^{3/0} = X^{3/0}, \\ \tau^3 \triangleright \hat{\partial}^\pm &= q^{\mp 4} \hat{\partial}^\pm, \quad \tau^3 \triangleright \hat{\partial}^0 = \hat{\partial}^0, \quad \tau^3 \triangleright \hat{\partial}^{3/0} = \hat{\partial}^{3/0}, \end{aligned} \quad (79)$$

together with the identities in (76) and (77) establishes that the exponentials have to take the form⁸

$$\begin{aligned} & \exp(x_R | \hat{\partial}_{\hat{L}}) \\ &= \sum_{\underline{m}=0}^{\infty} f^{\underline{m}}(\underline{X}) \otimes (\hat{\partial}^+)^{m_-} (\hat{\partial}^{3/0})^{m_3} (\hat{\partial}^0)^{m_{3/0}} (\hat{\partial}^-)^{m_+}, \end{aligned} \quad (80)$$

where

$$\begin{aligned} & f^{\underline{m}}(\underline{X}) \\ &= \sum_{\substack{2l+v \leq m_3 \\ -m_{3/0} \leq v, -m_\pm \leq l}} f_{l,v}^{\underline{m}} \cdot (X^+)^{m_++l} (X^{3/0})^{m_{3/0}+v} \\ & \quad \times (X^3)^{m_3-2l-v} (X^-)^{m_-+l}. \end{aligned} \quad (81)$$

In the following it is our aim to determine the unknown coefficients $f_{l,v}^{\underline{m}}$. Before doing this let us introduce, for brevity,

$$\begin{aligned} \hat{\partial}^{\underline{k}} &\equiv (\hat{\partial}^+)^{k_-} (\hat{\partial}^{3/0})^{k_3} (\hat{\partial}^0)^{k_{3/0}} (\hat{\partial}^-)^{k_+}, \\ X^{\underline{k}} &\equiv (X^+)^{k_+} (X^{3/0})^{k_{3/0}} (X^3)^{k_3} (X^-)^{k_-}. \end{aligned} \quad (82)$$

⁸ For notational convenience, we introduce a multi-index $\underline{m} \equiv (m_+, m_{3/0}, m_3, m_-)$.

Inserting the expressions of (80) and (81) into⁹

$$(\varepsilon \otimes \text{id}) \circ (\hat{\partial}^{\underline{k}} \otimes \text{id}) \triangleright \exp(x_R | \hat{\partial}_{\underline{L}}) = \hat{\partial}^{\underline{k}} \quad (83)$$

provides us with a system of equations given by

$$\sum_{\substack{2l+v \leq m_3 \\ -m_3/0 \leq v, -\min(m_+, m_-) \leq l}} f_{l,v}^{\underline{m}} \cdot \left\langle \hat{\partial}^{\underline{k}}, X^{\underline{m}+(l,v,-2l-v,l)} \right\rangle_{\hat{L},R} = \delta_{\underline{m},\underline{k}}, \quad (84)$$

where

$$\delta_{\underline{m},\underline{k}} = \delta_{m_+,k_+} \delta_{m_3,k_3} \delta_{m_3/0,k_3/0} \delta_{m_-,k_-}. \quad (85)$$

This system for the unknown coefficients $f_{l,v}^{\underline{m}}$ can be simplified further by taking the relations

$$\left\langle \hat{\partial}^{\underline{m}}, X^{\underline{m}+(l,v,-2l-v,l)} \right\rangle_{\hat{L},R} = 0, \quad \text{if } 2l+v > 0 \text{ or } v > 0, \quad (86)$$

a proof of which is given in Appendix B. By exploiting the property (86) one can then show that we have

$$f_{l,v}^{\underline{m}} = 0, \quad \text{if } v < 0 \text{ or } 2l+v < 0. \quad (87)$$

The proof of this assumption can again be found in Appendix B. Finally, a little thought using (86) and (87) shows that the system (84) can be reduced to

$$\sum_{\substack{0 \leq 2l+v \leq m_3 \\ 0 \leq v, -\min(m_+, m_-) \leq l}} f_{l,v}^{\underline{m}} \cdot \left\langle \hat{\partial}^{\underline{m}+(l',v',-2l'-v',l')}, X^{\underline{m}+(l,v,-2l-v,l)} \right\rangle_{\hat{L},R} = \delta_0^{v'} \delta_0^{l'}, \quad (88)$$

if \underline{k} is specified according to

$$\begin{aligned} \underline{k} &= \underline{m} + (l', v', -2l' - v', l') \\ &= (m_+ + l', m_3/0 + v', m_3 - 2l' - v', m_- + l'), \end{aligned} \quad (89)$$

where l' and v' are non-negative integers with

$$2l' + v' \leq m_3. \quad (90)$$

It is our next goal to present a method for solving the above system of equations. Towards this end we introduce the function

$$z(v, l) \equiv v + l + \left\lfloor \frac{v}{2} \right\rfloor + 1 + \sum_{i=0}^{v-1} \left(\left\lfloor \frac{m_3 - i}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor \right), \quad (91)$$

where $[s]$ denotes the biggest integer not being bigger than s . From the constraints on the summations in (84) and (88) we know that the integer values the variables v and l shall take on are restricted to

$$\begin{aligned} 0 &\leq v \leq v_{\max}(\underline{m}), \\ -\left\lfloor \frac{v}{2} \right\rfloor &\leq l \leq \left\lfloor \frac{m_3 - v}{2} \right\rfloor, \end{aligned} \quad (92)$$

where we have set $v_{\max}(\underline{m}) \equiv m_3 + 2 \min(m_+, m_-)$. For a better understanding of the following considerations it is important to notice that $z(v, l)$ shows the property

$$z(v, l) < z(v', l') \Leftrightarrow \begin{cases} v < v', \\ v = v', \quad l < l'. \end{cases} \quad (93)$$

This implies that the maximum value of $z(v, l)$ is given by

$$z_{\max}(\underline{m}) \equiv v_{\max}(\underline{m}) - \min(m_+, m_-) + \left\lfloor \frac{v_{\max}(\underline{m})}{2} \right\rfloor + 1 \quad (94)$$

$$+ \sum_{i=0}^{v_{\max}(\underline{m})} \left(\left\lfloor \frac{m_3 - i}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor \right). \quad (95)$$

It is not very difficult to convince oneself that we can also establish a one-to-one correspondence between the allowed values of v and l on the one hand and those of $z(v, l)$ on the other hand by setting

$$\begin{aligned} v_z(m_3) &\equiv \max \left\{ j \in \mathbb{N}_0 \mid z - j - \sum_{i=0}^{j-1} \left(\left\lfloor \frac{m_3 - i}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor \right) > 0 \right\}, \\ l_z(m_3) &\equiv z - v_z(m_3) - \left\lfloor \frac{v_z(m_3)}{2} \right\rfloor - 1 \\ &\quad - \sum_{i=0}^{v_z(m_3)-1} \left(\left\lfloor \frac{m_3 - i}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor \right). \end{aligned} \quad (96)$$

The deeper reason for introducing the function $z(v, l)$ becomes quite clear, as soon as one realizes that it establishes an ordering for the coefficients $f_{l,v}^{\underline{m}}$ if we take the convention

$$f_{z(v,l)}^{\underline{m}} \equiv f_{v,l}^{\underline{m}}. \quad (97)$$

By using this ordering the system (88) can be rewritten as

$$\sum_{j=1}^{z_{\max}(\underline{m})} \Theta(l_j + \min(m_+, m_-)) \cdot f_j^{\underline{m}} \cdot \langle z(v', l'), j \rangle_{\hat{L},R}^{\underline{m}} = \delta_0^{v'} \delta_0^{l'}, \quad (98)$$

where we have introduced the step-function

$$\Theta(h) = \begin{cases} 0, & \text{if } h < 0, \\ 1, & \text{otherwise,} \end{cases} \quad (99)$$

and as a short-hand notation¹⁰

$$\begin{aligned} \langle k, j \rangle_{\hat{L},R}^{\underline{m}} &\equiv \left\langle \hat{\partial}^{\underline{m}+(l_k, v_k, -2l_k - v_k, l_k)}, X^{\underline{m}+(l_j, v_j, -2l_j - v_j, l_j)} \right\rangle_{\hat{L},R}. \end{aligned} \quad (100)$$

⁹ This formula follows from a direct application of (13) and (15).

¹⁰ To understand the equivalence of (88) and (98) it is helpful to keep in mind that $v_{z(v',l')} = v'$ and $l_{z(v',l')} = l'$.

Generalizing relation (86) to¹¹

$$\langle k, j \rangle_{\hat{L}, R}^m = 0, \quad \text{if } k < j, \quad (101)$$

shows us that it is sufficient to choose $z(v', l')$ as an upper bound of the sum over j , i.e.

$$\sum_{j=1}^{z(v', l')} \Theta(l_j + \min(m_+, m_-)) \cdot f_j^m \cdot \langle z(v', l'), j \rangle_{\hat{L}, R}^m = \delta_0^{v'} \delta_0^{l'} \quad (102)$$

or

$$\sum_{j=1}^k \Theta(l_j + \min(m_+, m_-)) \cdot f_j^m \cdot \langle k, j \rangle_{\hat{L}, R}^m = \delta_0^{v_k} \delta_0^{l_k}, \quad (103)$$

where $1 \leq k \leq z_{\max}(\underline{m})$. In this way we have arrived at a system of triangular form which we can reduce to the recursion relation

$$f_k^m = - \sum_{1 \leq j < k} \Theta(l_j + \min(m_+, m_-)) \times f_j^m \cdot \frac{\langle k, j \rangle_{\hat{L}, R}^m}{\langle k, k \rangle_{\hat{L}, R}^m}, \quad \text{for } 1 < k \leq z_{\max}(\underline{m}), \quad (104)$$

$$f_1^m = \frac{1}{\langle 1, 1 \rangle_{\hat{L}, R}^m} = \langle \hat{\partial}^m, X^m \rangle_{\hat{L}, R} = \frac{(-q)^{m_- - m_+}}{[[m_-]]_{q^2}! [[m_{3/0}]]_{q^2}! [[m_3]]_{q^2}! [[m_+]_{q^2}!}. \quad (105)$$

Notice that the last expression for f_1^m in (105) can be derived rather easily by multiple application of the representations presented in [6]. Now, it should be obvious that the coefficients f_k^m can be expressed as

$$f_k^m = \sum_{i=1}^{k-1} \sum_{1=j_0 < j_1 < \dots < j_i=k} (-1)^i \times \frac{\prod_{p=1}^i \Theta(l_{j_p} + \min(m_+, m_-)) \cdot \langle j_p, j_{p-1} \rangle_{\hat{L}, R}^m}{\prod_{r=0}^i \langle j_r, j_r \rangle_{\hat{L}, R}^m}. \quad (106)$$

Substituting this into (81) together with (80) finally yields

$$\exp(x_R | \hat{\partial}_{\hat{L}}) = \sum_{\underline{m}=0}^{\infty} \sum_{k=1}^{z_{\max}(\underline{m})} (C)_k^m \cdot \frac{(-q)^{m_- - m_+} X^{\underline{m} + (l_k, v_k, -2l_k - v_k, l_k)} \otimes \hat{\partial}^{\underline{m}}}{[[m_-]]_{q^2}! [[m_{3/0}]]_{q^2}! [[m_3]]_{q^2}! [[m_+]_{q^2}!}, \quad (107)$$

where

¹¹ For a proof of this assumption see Appendix B.

$$(C)_k^m = \sum_{i=1}^{k-1} \sum_{1=j_0 < j_1 < \dots < j_i=k} (-1)^i \times \prod_{p=1}^i \frac{\Theta(l_{j_p} + \min(m_+, m_-)) \cdot \langle j_p, j_{p-1} \rangle_{\hat{L}, R}^m}{\langle j_r, j_r \rangle_{\hat{L}, R}^m}. \quad (108)$$

Taking into account the conjugation properties [38]

$$\overline{\sum_a f^a \otimes e_a} = \sum_a \overline{e_a} \otimes \overline{f^a} \quad (109)$$

and

$$\overline{X^0} = X^0, \quad \overline{X^{3/0}} = X^{3/0}, \quad \overline{X^{\pm}} = -q^{\mp 1} X^{\mp}, \\ \overline{\hat{\partial}^0} = \hat{\partial}^0, \quad \overline{\hat{\partial}^{3/0}} = \hat{\partial}^{3/0}, \quad \overline{\hat{\partial}^{\pm}} = -q^{\mp 1} \hat{\partial}^{\mp}, \quad (110)$$

immediately gives us

$$\exp(\hat{\partial}_R | x_{\hat{L}}) = \overline{\exp(x_R | \hat{\partial}_{\hat{L}})} = \sum_{\underline{m}=0}^{\infty} \sum_{k=1}^{z_{\max}(\underline{m}_3)} (\bar{C})_k^m \cdot \frac{(-q)^{m_+ - m_-} \hat{\partial}^{\underline{m}} \otimes X^{\underline{m} + (l_k, v_k, -2l_k - v_k, l_k)}}{[[m_-]]_{q^2}! [[m_{3/0}]]_{q^2}! [[m_3]]_{q^2}! [[m_+]_{q^2}!}, \quad (111)$$

where

$$(\bar{C})_k^m = \sum_{i=1}^{k-1} \sum_{1=j_0 < j_1 < \dots < j_i=k} (-1)^i \times \prod_{p=1}^i \frac{\Theta(l_{j_p} + \min(m_+, m_-)) \cdot \langle j_{p-1}, j_p \rangle_{\hat{L}, R}^m}{\langle j_r, j_r \rangle_{\hat{L}, R}^m} \quad (112)$$

with

$$\overline{\langle k, j \rangle_{\hat{L}, R}^m} \equiv \left\langle X^{\underline{m} + (l_j, v_j, -2l_j - v_j, l_j)}, \hat{\partial}^{\underline{m} + (l_k, v_k, -2l_k - v_k, l_k)} \right\rangle_{\hat{L}, R}. \quad (113)$$

In this sense, we have found expressions for q -exponentials in terms of the dual pairing between coordinates and derivatives. It remains to evaluate the expressions in (100) and (113). But this is a rather tedious task which has to be done elsewhere. Thus, we do not want to discuss that issue any further here.

For completeness we wish to present the rules making a connection between the different types of q -exponentials. With the same reasonings already applied to the Euclidean cases we can now write

$$\exp(x_R | \hat{\partial}_{\hat{L}}) \stackrel{\pm \rightarrow \mp}{q \leftarrow 1/q} \widetilde{\exp}(x_{\hat{R}} | \partial_L), \\ \exp(\hat{\partial}_R | x_{\hat{L}}) \stackrel{\pm \rightarrow \mp}{q \leftarrow 1/q} \widetilde{\exp}(\partial_{\hat{R}} | x_L), \quad (114)$$

where $\begin{matrix} \pm \rightarrow \mp \\ q \rightarrow 1/q \end{matrix}$ symbolizes the substitutions

$$\begin{aligned} q &\leftrightarrow q^{-1}, & X^\pm &\leftrightarrow X^\mp, \\ \hat{\partial}^\pm &\leftrightarrow \partial^\mp, & \hat{\partial}^{3/0} &\leftrightarrow \partial^{3/0}, & \hat{\partial}^0 &\leftrightarrow \partial^0. \end{aligned} \quad (115)$$

6 Remarks

Let us end with a few comments on the explicit expressions we have derived for q -exponentials. In the Euclidean cases the formulae for dual pairing and q -exponential are in complete analogy to those of their classical counterparts. In particular, they do not depend on terms proportional to powers of the deformation parameter $\lambda = q - q^{-1}$. In the case of q -deformed Minkowski space the situation is a little bit different, as it is impossible to find two sets of normally ordered monomials which constitute two bases being dual to each other. In other words, for every choice of normally ordered monomials there are terms like

$$\langle \hat{\partial}^- \hat{\partial}^+, (X^3)^2 \rangle_{L,R} = \lambda(1 + q\lambda_+^{-1}), \quad \lambda_+ = q + q^{-1}, \quad (116)$$

vanishing in the undeformed limit as $q \rightarrow 1$. It is for this reason that in expression (107) non-classical factors $(C)_k^m$ appear which depend on powers of λ in such a way that

$$(C)_k^m \xrightarrow{q \rightarrow 1} \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (117)$$

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A Quantum spaces

In this appendix we list for the quantum spaces under consideration the explicit form of their defining commutation relations, their conjugation properties and the non-vanishing elements of the quantum metric.

The coordinates of 2-dimensional q -deformed Euclidean space fulfill the relation

$$X^1 X^2 = q X^2 X^1, \quad (118)$$

whereas the quantum metric is given by a matrix ε^{ij} with non-vanishing elements

$$\varepsilon^{12} = q^{-1/2}, \quad \varepsilon^{21} = -q^{1/2}. \quad (119)$$

Furthermore, the relation (118) is compatible with the conjugation assignment:

$$\overline{X^i} = -\varepsilon_{ij} X^j \quad (120)$$

where ε_{ij} denotes the inverse of ε^{ij} .

In the case of q -deformed Euclidean space in three dimensions the commutation relations read

$$\begin{aligned} X^3 X^+ &= q^2 X^+ X^3, \\ X^- X^3 &= q^2 X^3 X^-, \\ X^- X^+ &= X^+ X^- + \lambda X^3 X^3. \end{aligned} \quad (121)$$

The non-vanishing elements of the quantum metric are

$$g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -q^{-1}. \quad (122)$$

Now, the conjugation of the coordinates is given by

$$\overline{X^A} = g_{AB} X^B \quad (123)$$

with g_{AB} being the inverse of g^{AB} .

For the 4-dimensional Euclidean space, we have the relations

$$\begin{aligned} X^1 X^2 &= q X^2 X^1, \\ X^1 X^3 &= q X^3 X^1, \\ X^3 X^4 &= q X^4 X^3, \\ X^2 X^4 &= q X^4 X^2, \\ X^2 X^3 &= X^3 X^2, \\ X^4 X^1 &= X^1 X^4 + \lambda X^2 X^3. \end{aligned} \quad (124)$$

The metric has the non-vanishing components

$$g^{14} = q^{-1}, \quad g^{23} = g^{32} = 1, \quad g^{41} = q. \quad (126)$$

The inverse g_{ij} of this metric can again be used to formulate the conjugation properties for the coordinates, i.e.

$$\overline{X^i} = g_{ij} X^j. \quad (127)$$

For q -deformed Minkowski space one has the relations

$$\begin{aligned} X^\mu X^0 &= X^0 X^\mu, \quad \mu \in \{0, +, -, 3\}, \\ X^- X^3 - q^2 X^3 X^- &= -q\lambda X^0 X^-, \\ X^3 X^+ - q^2 X^+ X^3 &= -q\lambda X^0 X^+, \\ X^- X^+ - X^+ X^- &= \lambda(X^3 X^3 - X^0 X^3), \end{aligned} \quad (128)$$

and the metric

$$g^{00} = -1, \quad g^{33} = 1, \quad g^{+-} = -q, \quad g^{-+} = -q^{-1}. \quad (129)$$

Finally, the conjugation on q -deformed Minkowski space is determined by

$$\overline{X^0} = X^0, \quad \overline{X^3} = X^3, \quad \overline{X^\pm} = -q^\mp X^\mp. \quad (130)$$

B Proofs

(1) *The Proof of (86) and (101).*

Recalling that $\hat{\partial}^{3/0}$ obeys the commutation relations

$$\hat{\partial}^{3/0} \hat{\partial}^0 = \hat{\partial}^0 \hat{\partial}^{3/0}, \quad \hat{\partial}^{3/0} \hat{\partial}^- = q^{-2} \hat{\partial}^- \hat{\partial}^{3/0} \quad (131)$$

and

$$\hat{\partial}^{3/0} X^+ = X^+ \hat{\partial}^{3/0}, \quad \hat{\partial}^{3/0} X^{3/0} = X^{3/0} \hat{\partial}^{3/0}, \quad (132)$$

we can rewrite the dual pairing as follows:

$$\begin{aligned} & \left\langle \hat{\partial}^{\underline{m}}, X^{\underline{m}+(l,v,-2l-v,l)} \right\rangle_{\hat{L},R} \\ &= q^{2lm_3} \left[(\hat{\partial}^+)^{m_-} (\hat{\partial}^0)^{m_{3/0}} (\hat{\partial}^-)^{m_+} \bar{\varepsilon} (X^+)^{m_+ + l} (X^3)^{m_{3/0} + v} \right. \\ & \quad \left. \cdot \left((\hat{\partial}^{3/0})^{m_3} \bar{\varepsilon} (X^3)^{m_3 - 2l - v} (X^-)^{m_- + l} \right) \right] \Big|_{\underline{X}=0}. \end{aligned} \quad (133)$$

Direct inspection of the representations of $\hat{\partial}^{3/0}$ (presented in [6]) shows that

$$(\hat{\partial}^{3/0})^n \bar{\varepsilon} (X^3)^{k_3} (X^-)^{k_-} = 0, \quad \text{if } n > k_3, \quad (134)$$

which, in turn, tells us that the last expression in (133) has to vanish, if $2l + v > 0$.

Now we come to the case $v > 0$. First of all, let us note that the roles of coordinates and derivatives can be completely reversed. In this sense we proceed as follows:

$$\begin{aligned} & \left\langle \hat{\partial}^{\underline{m}}, X^{\underline{m}+(l,v,-2l-v,l)} \right\rangle_{\hat{L},R} \\ &= (\hat{\partial}^{\underline{m}} \bar{\varepsilon} X^{\underline{m}+(l,v,-2l-v,l)}) \Big|_{\underline{X}=0} \\ &= (\hat{\partial}^{\underline{m}} \triangleleft X^{\underline{m}+(l,v,-2l-v,l)}) \Big|_{\hat{\partial}=0} \\ &= (X^{\underline{m}+(l,v,-2l-v,l)} \bar{\varepsilon} \hat{\partial}^{\underline{m}}) \Big|_{\hat{\partial}=0}, \end{aligned} \quad (135)$$

where we used for the last identity that

$$\varepsilon(f) = \overline{\varepsilon(f)} = \varepsilon(\bar{f}). \quad (136)$$

Applying the formulae

$$\begin{aligned} (X^3)^n &= (X^{3/0} + X^0)^n \\ &= \sum_{i=0}^n \binom{n}{i} (X^{3/0})^i (X^0)^{n-i}, \end{aligned} \quad (137)$$

$$\begin{aligned} (\hat{\partial}^0)^m &= (\hat{\partial}^3 - \hat{\partial}^{3/0})^m \\ &= \sum_{j=0}^m \binom{m}{j} (\hat{\partial}^3)^{m-j} (-\hat{\partial}^{3/0})^j \end{aligned} \quad (138)$$

along with

$$\begin{aligned} & \left[(X^+)^{n_+} (X^{3/0})^{n_{3/0}} (X^0)^{n_0} (X^-)^{n_-} \bar{\varepsilon} \right. \\ & \quad \left. (\hat{\partial}^+)^{m_-} (\hat{\partial}^{3/0})^{m_{3/0}} (\hat{\partial}^3)^{m_3} (\hat{\partial}^-)^{m_+} \right] \Big|_{\hat{\partial}=0} \\ &= \left[(\hat{\partial}^+)^{n_+} (\hat{\partial}^{3/0})^{n_{3/0}} (\hat{\partial}^0)^{n_0} (\hat{\partial}^-)^{n_-} \bar{\varepsilon} \right. \\ & \quad \left. (X^+)^{m_-} (X^{3/0})^{m_{3/0}} (X^3)^{m_3} (X^-)^{m_+} \right] \Big|_{\underline{X}=0} \end{aligned} \quad (139)$$

to the last expression in (135) gives us the identity

$$\begin{aligned} & \left\langle \hat{\partial}^{\underline{m}}, X^{\underline{m}+(l,v,-2l-v,l)} \right\rangle_{\hat{L},R} \\ &= \sum_{i=0}^{m_3-2l-v} \sum_{j=0}^{m_{3/0}} (-1)^j \binom{m_3-2l-v}{i} \binom{m_{3/0}}{j} \\ & \quad \times \left[(\hat{\partial}^+)^{m_-+l} (\hat{\partial}^{3/0})^{m_{3/0}+v+i} (\hat{\partial}^0)^{m_3-2l-v-i} (\hat{\partial}^-)^{m_+} \bar{\varepsilon} \right. \\ & \quad \left. (X^+)^{m_+} (X^{3/0})^{m_3+j} (X^3)^{m_{3/0}-j} (X^-)^{m_-} \right] \Big|_{\underline{X}=0}. \end{aligned} \quad (140)$$

Since in the case $v > 0$ the exponent of $\hat{\partial}^{3/0}$ is strictly greater than the exponent of X^3 it follows from (134) that

$$\left\langle \hat{\partial}^{\underline{m}}, X^{\underline{m}+(l,v,-2l-v,l)} \right\rangle_{\hat{L},R} = 0, \quad \text{if } v > 0. \quad (141)$$

Additionally, we get from the above results

$$\begin{aligned} & \langle k, j \rangle_{\hat{L},R}^{\underline{m}} \\ &= \left\langle \hat{\partial}^{\underline{m}+(l_k, v_k, -2l_k - v_k, l_k)}, X^{\underline{m}+(l_j, v_j, -2l_j - v_j, l_j)} \right\rangle_{\hat{L},R} \\ &= \left\langle \hat{\partial}^{\underline{m}'}, X^{\underline{m}'+(l_j - l_k, v_j - v_k, -2(l_j - l_k) - (v_j - v_k), l_j - l_k)} \right\rangle_{\hat{L},R} \\ &= 0, \quad \text{if } v_j > v_k \text{ or } 2l_j + v_j > 2l_k + v_k, \end{aligned} \quad (142)$$

where $\underline{m}' = \underline{m} + (l_k, v_k, -2l_k - v_k, l_k)$. However, the condition in (142) holds for $k < j$ which immediately gives us

$$\langle k, j \rangle_{\hat{L},R}^{\underline{m}} = 0, \quad \text{if } k < j. \quad (143)$$

(2) *The Proof of (87).*

By specifying the multi-index \underline{k} to

$$\underline{k} = \underline{m} + (l', v', -2l' - v', l') \quad (144)$$

with

$$\begin{aligned} -\min(m_+, m_-) &\leq l', \\ -m_{3/0} &\leq v' \leq -1, \\ 2l' + v' &\leq m_3, \end{aligned} \quad (145)$$

the system (84) reduces to

$$\begin{aligned} & \sum_{\substack{-m_{3/0} \leq v \leq v', -\min(m_+, m_-) \leq l \\ 2l+v \leq 2l'+v'}} f_{l,v}^{\underline{m}} \\ & \quad \times \left\langle \hat{\partial}^{\underline{m}'}, X^{\underline{m}'+(l-l', v-v', -2(l-l')-(v-v'), l-l')} \right\rangle_{\hat{L},R} \\ &= \delta_0^{v'} \delta_0^{l'}, \end{aligned} \quad (146)$$

where $\underline{m}' = \underline{m} + (l', v', -2l' - v', l')$. Notice that we have a one-to-one correspondence between the equations of the above subsystem and the pairs (l', v') . Arranging the equations of this system in an order determined by

$$(l'_1, v'_1) < (l'_2, v'_2) \Leftrightarrow \begin{cases} 2l'_1 + v'_1 < 2l'_2 + v'_2, \\ 2l'_1 + v'_1 = 2l'_2 + v'_2, & v'_1 < v'_2, \end{cases} \quad (147)$$

will then show us that it is of triangular form and solves for

$$f_{l,v}^{\underline{m}} = 0, \quad \text{if } 2l + v < 0 \text{ or } v < 0. \quad (148)$$

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